

# Kawamata-Viehweg Vanishing on Rational Surfaces in Positive Characteristic

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## Abstract

We prove that the Kawamata-Viehweg vanishing theorem holds on rational surfaces in positive characteristic by means of the lifting property to  $W_2(k)$  of certain log pairs on smooth rational surfaces. As a corollary, the Kawamata-Viehweg vanishing theorem holds on log del Pezzo surfaces in positive characteristic.

## 1 Introduction

There are many generalizations of the celebrated Kodaira vanishing theorem. One of the most important generalizations is the Kawamata-Viehweg vanishing theorem. As is well known, it is inevitable to run the higher dimensional minimal model program in the categories of varieties with suitable singularities, hence we have to consider  $\mathbb{Q}$ -divisors instead of integral divisors. It turns out that the Kawamata-Viehweg vanishing theorem is indispensable and plays a crucial role in birational geometry of higher dimensional algebraic varieties.

The Kawamata-Viehweg vanishing theorem is of several forms. The one dealing with ample  $\mathbb{Q}$ -divisors follows directly from the Kodaira vanishing theorem via the Kummer covering trick [Ka82, Vi82].

**Theorem 1.1** (Kawamata-Viehweg vanishing). *Let  $X$  be a smooth projective variety over an algebraically closed field  $k$  with  $\text{char}(k) = 0$ , and  $H$  an ample  $\mathbb{Q}$ -divisor on  $X$  such that the fractional part  $\langle H \rangle$  has simple normal crossing support. Then  $H^i(X, K_X + \lceil H \rceil) = 0$  holds for any  $i > 0$ .*

The most general form is stated for log pairs which have only Kawamata log terminal singularities [KMM87, Theorem 1-2-5].

**Theorem 1.2** (Kawamata-Viehweg vanishing). *Let  $X$  be a normal projective variety over an algebraically closed field  $k$  with  $\text{char}(k) = 0$ ,  $B = \sum b_i B_i$  an effective  $\mathbb{Q}$ -divisor on  $X$ , and  $D$  a  $\mathbb{Q}$ -Cartier Weil divisor on  $X$ . Assume that  $(X, B)$  is Kawamata log terminal (KLT for short), and  $D - (K_X + B)$  is ample. Then  $H^i(X, D) = 0$  holds for any  $i > 0$ .*

The original proof of the Kodaira vanishing theorem was analytic, and its purely algebraic proof was first given by Deligne and Illusie [DI87]. For a smooth proper variety  $X$  over a perfect field  $k$  of characteristic  $p > 0$ , they have defined the notion of a lifting of  $X$  to  $W_2(k)$ , the ring of Witt vectors of length two of  $k$ , and have proved that if  $X$  admits a lifting to  $W_2(k)$  and  $\dim X < p$ , then the de Rham complex is decomposable in derived category, consequently the Hodge to de Rham spectral

sequence degenerates in  $E_1$ , and the Kodaira-Akizuki-Nakano vanishing theorem holds on  $X$ . The characteristic zero case can be deduced from the characteristic  $p$  case by standard arguments [II96, §6]. Furthermore, they have also claimed the corresponding results [DI87, §4.2] in the logarithmic case for a log pair  $(X, D)$ , where  $X$  is a smooth proper variety and  $D \subset X$  is a simple normal crossing divisor over  $k$ . The explicit statements and proofs of those results were given by Esnault and Viehweg [EV92, §8-§11]. In particular, if  $(X, D)$  admits a lifting to  $W_2(k)$ , then the logarithmic Kodaira-Akizuki-Nakano vanishing theorem holds on  $X$ .

Later, Hara [Ha98] and Matsuki and Olsson [MO05] independently proved the Kawamata-Viehweg vanishing theorem in positive characteristic under the lifting condition to  $W_2(k)$  of certain log pairs. The method of Hara is to use a quasi-isomorphism between the logarithmic de Rham complex and its variant by adding certain modulo  $p$  fractional parts, while Matsuki and Olsson replaced the Kummer covering trick with the stack technique, which behaves well in arbitrary characteristic, and interpreted the Kawamata-Viehweg vanishing on varieties as the Kodaira vanishing on stacks.

**Theorem 1.3** (Kawamata-Viehweg vanishing in char.  $p > 0$ ). *Let  $k$  be a perfect field of characteristic  $p > 0$ ,  $X$  a smooth projective variety over  $k$  of dimension  $d$ ,  $H$  an ample  $\mathbb{Q}$ -divisor on  $X$ , and  $D$  a simple normal crossing divisor containing  $\text{Supp}(\langle H \rangle)$ . Assume that  $(X, D)$  admits a lifting to  $W_2(k)$ . Then*

$$H^i(X, \Omega_X^j(\log D)(-\lceil H \rceil)) = 0 \text{ holds for any } i + j < \inf(d, p).$$

*In particular,  $H^i(X, K_X + \lceil H \rceil) = 0$  holds for any  $i > d - \inf(d, p)$ .*

The lifting condition to  $W_2(k)$ , together with the reduction modulo  $p$  technique, is usually used to prove some statements in characteristic zero. However, the lifting condition is indeed a very strong condition, since it is not satisfied even for some log pairs with simple structure (see Corollary 1.10).

In what follows, we always work over an algebraically closed field  $k$  of characteristic  $p > 0$  unless otherwise stated. The following main theorem, i.e. the Kawamata-Viehweg vanishing theorem on rational surfaces, will be proved in this paper.

**Theorem 1.4.** *Let  $X$  be a normal projective rational surface,  $D$  a  $\mathbb{Q}$ -Cartier Weil divisor on  $X$ , and  $B$  an effective  $\mathbb{Q}$ -divisor such that  $(X, B)$  is KLT and  $D - (K_X + B)$  is ample. Then  $H^1(X, D) = 0$  holds.*

Thanks to Theorem 1.3, we have only to verify that the lifting condition to  $W_2(k)$  holds for certain log pairs on smooth rational surfaces. The main idea of the proof is to reduce the problem to the Hirzebruch surface case.

**Definition 1.5.** A pair  $(X, B)$  is called a log del Pezzo surface, if  $X$  is a normal projective surface, and  $B$  is an effective  $\mathbb{Q}$ -divisor on  $X$  such that  $(X, B)$  is KLT and  $-(K_X + B)$  is ample.

A normal projective surface  $X$  is called a log del Pezzo surface (resp. weak log del Pezzo surface), if  $(X, 0)$  is KLT and  $-K_X$  is ample (resp. nef and big).

There are some corollaries of the main theorem.

**Corollary 1.6.** *Let  $(X, B)$  be a log del Pezzo surface,  $D$  a  $\mathbb{Q}$ -Cartier Weil divisor on  $X$  such that  $D - (K_X + B)$  is ample. Then  $H^1(X, D) = 0$  holds.*

**Corollary 1.7.** *Let  $X$  be a (weak) log del Pezzo surface. Then  $H^1(X, \mathcal{O}_X) = 0$  holds.*

*Remark 1.8.* A Fano variety, by definition, is a projective variety  $X$  with the anti-canonical divisor  $-K_X$  ample. Fano surface is conventionally called del Pezzo surface. As is well known, Fano variety has appeared as a kind of outcome of running the minimal model program, so the study of Fano varieties is of certain interest in birational geometry of algebraic varieties. Let us recall some known vanishing or non-vanishing results concerning Fano varieties in positive characteristic, which show that Corollary 1.6 is just a result as expected.

(1) Tango [Ta72] has proved that the Kodaira vanishing theorem does hold on smooth projective ruled surfaces, hence on *smooth* del Pezzo surface.

(2) Reid [Re94] has found *nonnormal* del Pezzo surfaces  $X$  with  $H^1(X, \mathcal{O}_X) \neq 0$ .

(3) Schröer [Sc07] proved that over any *nonperfect* field  $k$  of characteristic  $p = 2$ , there is a normal del Pezzo surface  $X$  with  $H^1(X, \mathcal{O}_X) \neq 0$ .

(4) Shepherd-Barron [SB97] established that  $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$  holds for smooth Fano *threefolds*.

(5) Lauritzen and Rao [LR97] has constructed counterexamples to the Kodaira vanishing theorem on some smooth Fano varieties of *dimension at least 6*.

Theorem 1.4 also implies the following corollary, which is a weak version of the logarithmic Kollár vanishing theorem [Ko95, Theorem 10.19] and the logarithmic semi-positivity theorem [Ka00, Theorem 1.2 and Corollary 1.3] on rational surfaces.

**Corollary 1.9.** *Let  $X$  be a normal projective rational surface,  $f : X \rightarrow \mathbb{P}^1$  a surjective proper morphism, and  $B$  an effective  $\mathbb{Q}$ -divisor on  $X$  such that  $(X, B)$  is KLT. Let  $D$  be a  $\mathbb{Q}$ -Cartier Weil divisor on  $X$  such that  $D - (K_X + B)$  is ample. Then*

- (1)  $H^1(\mathbb{P}^1, R^i f_* \mathcal{O}_X(D)) = 0$  holds for any  $i \geq 0$ , and
- (2)  $f_* \mathcal{O}_X(D - f^* K_{\mathbb{P}^1})$  is an ample vector bundle on  $\mathbb{P}^1$ .

Unfortunately, Theorem 1.4 and Corollary 1.9 fail for certain ruled surfaces (see [Xie06, Examples 3.7, 3.9, 3.10]). As a consequence, it follows that the lifting condition to  $W_2(k)$  is not satisfied even for some log pairs on geometrically ruled surfaces (see [Xie07, Definition 2.6] for the definition of Tango curve).

**Corollary 1.10.** *If  $C$  is a Tango curve, then there are a  $\mathbb{P}^1$ -bundle  $f : X \rightarrow C$  and a smooth curve  $C' \subset X$  such that  $(X, C')$  cannot be lifted to  $W_2(k)$ .*

In §2, we will prove some results concerning the lifting property of certain log pairs on smooth rational surfaces. §3 is devoted to the proofs of the main theorem and the corollaries. Finally, we will give some remarks on the main results in §4. For the necessary notions and results in birational geometry, e.g. Kawamata log terminal singularity, we refer the reader to [KMM87] and [KM98].

**Notation.** We use  $\equiv$  to denote numerical equivalence, and  $[B] = \sum [b_i] B_i$  (resp.  $\lceil B \rceil = \sum \lceil b_i \rceil B_i$ ,  $\langle B \rangle = \sum \langle b_i \rangle B_i$ ,  $\{B\} = \sum \{b_i\} B_i$ ) to denote the round-down (resp. round-up, fractional part, upper fractional part) of a  $\mathbb{Q}$ -divisor  $B = \sum b_i B_i$ , where for a real number  $b$ ,  $[b] := \max\{n \in \mathbb{Z} \mid n \leq b\}$ ,  $\lceil b \rceil := -[-b]$ ,  $\langle b \rangle := b - [b]$  and  $\{b\} := \lceil b \rceil - b$ .

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## 2 Lifting property on smooth rational surfaces

Let us first recall some definitions from [EV92, Definition 8.11].

**Definition 2.1.** Let  $W_2(k)$  be the ring of Witt vectors of length two of  $k$ . Then  $W_2(k)$  is flat over  $\mathbb{Z}/p^2\mathbb{Z}$ , and  $W_2(k) \otimes_{\mathbb{Z}/p^2\mathbb{Z}} \mathbb{F}_p = k$ . For the explicit construction and further properties of  $W_2(k)$ , we refer the reader to [Se62, II.6]. The following definition generalizes the definition [DI87, 1.6] of liftings of  $k$ -schemes to  $W_2(k)$ .

Let  $X$  be a noetherian scheme over  $k$ , and  $D = \sum D_i$  a reduced Cartier divisor on  $X$ . A lifting of  $(X, D)$  to  $W_2(k)$  consists of a scheme  $\tilde{X}$  and closed subschemes  $\tilde{D}_i \subset \tilde{X}$ , all defined and flat over  $W_2(k)$  such that  $X = \tilde{X} \times_{\text{Spec } W_2(k)} \text{Spec } k$  and  $D_i = \tilde{D}_i \times_{\text{Spec } W_2(k)} \text{Spec } k$ . We write  $\tilde{D} = \sum \tilde{D}_i$  and say that  $(\tilde{X}, \tilde{D})$  is a lifting of  $(X, D)$  to  $W_2(k)$ , if no confusion is likely.

In the above definition, assume further that  $X$  is smooth over  $k$  and  $D = \sum D_i$  is simple normal crossing. If  $(\tilde{X}, \tilde{D})$  is a lifting of  $(X, D)$  to  $W_2(k)$ , then  $\tilde{X}$  is smooth over  $W_2(k)$  and  $\tilde{D} = \sum \tilde{D}_i$  is simple normal crossing over  $W_2(k)$ , i.e.  $\tilde{X}$  is covered by affine open subsets  $\{U\}$ , such that each  $U$  is étale over  $\mathbb{A}_{W_2(k)}^n$  via coordinates  $\{x_1, \dots, x_n\}$  and  $\tilde{D}|_U$  is defined by the equation  $x_1 \cdots x_\nu = 0$  with  $1 \leq \nu \leq n$  (see [EV92, Lemmas 8.13, 8.14]).

If  $\tilde{X}$  is a lifting of  $X$  to  $W_2(k)$ , then there is an exact sequence of  $\mathcal{O}_{\tilde{X}}$ -modules

$$0 \rightarrow \mathcal{O}_X \xrightarrow{p} \mathcal{O}_{\tilde{X}} \xrightarrow{r} \mathcal{O}_X \rightarrow 0,$$

where  $p(x) := px$  and  $r(\tilde{x}) := \tilde{x} \bmod p$  for  $x \in \mathcal{O}_X, \tilde{x} \in \mathcal{O}_{\tilde{X}}$  (see [EV92, Lemma 8.13]).

For instance,  $\mathbb{A}_k^n, \mathbb{P}_k^n$  and  $H_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(-m))$  have liftings to  $W_2(k)$ .

**Definition 2.2.** Let  $X$  be a smooth scheme over  $k$ ,  $D = \sum D_i$  a reduced divisor on  $X$ , and  $Z$  a closed subscheme of  $X$  smooth over  $k$  of codimension  $s \geq 2$ . A mixed lifting of  $(X, D + Z)$  to  $W_2(k)$  consists of a smooth scheme  $\tilde{X}$  over  $W_2(k)$ , closed subschemes  $\tilde{D}_i \subset \tilde{X}$  flat over  $W_2(k)$ , and a closed subscheme  $\tilde{Z} \subset \tilde{X}$  smooth over  $W_2(k)$  such that  $X = \tilde{X} \times_{\text{Spec } W_2(k)} \text{Spec } k$ ,  $D_i = \tilde{D}_i \times_{\text{Spec } W_2(k)} \text{Spec } k$  and  $Z = \tilde{Z} \times_{\text{Spec } W_2(k)} \text{Spec } k$ . We write  $\tilde{D} = \sum \tilde{D}_i$  and say that  $(\tilde{X}, \tilde{D} + \tilde{Z})$  is a mixed lifting of  $(X, D + Z)$  to  $W_2(k)$ , if no confusion is likely.

In the above definition, either  $D = \emptyset$  or  $Z = \emptyset$  is allowed. Obviously, if  $Z = \emptyset$  then a mixed lifting  $(\tilde{X}, \tilde{D})$  of  $(X, D)$  is indeed a lifting of  $(X, D)$  to  $W_2(k)$ .

For instance, if  $X = \mathbb{A}_k^n$  or  $\mathbb{P}_k^n$  or  $H_m$ , and  $P \in X$  is a closed point (or an infinitesimal closed point), then  $(X, P)$  has a mixed lifting to  $W_2(k)$ .

We need the following elementary lemmas.

**Lemma 2.3.** *Let  $X = H_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(-m))$  be a Hirzebruch surface with  $m \geq 0$ . Then for any reduced divisor  $D$  on  $X$ ,  $(X, D)$  has a mixed lifting to  $W_2(k)$ .*

*Proof.* Since  $X$  has a natural lifting  $\tilde{X} = \widetilde{H_m} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_{W_2(k)}} \oplus \mathcal{O}_{\mathbb{P}^1_{W_2(k)}}(-m))$ , we have only to lift the irreducible components of  $D$  to  $W_2(k)$  one by one. Thus we may assume that  $D$  is irreducible. Let  $f : X \rightarrow \mathbb{P}^1$  be the natural projection. Take a section  $E$  of  $f$  with  $\mathcal{O}_X(E) \cong \mathcal{O}_X(1)$  and  $E^2 = -m \leq 0$ .

If  $D.E < 0$  then we have  $D = E$  and  $E^2 < 0$ . In this case,  $D$  has a lifting  $\tilde{D}$ , which is the unique curve on  $\tilde{X}$  with negative self-intersection.

From now on, assume  $D.E \geq 0$ . The following exact sequence of abelian sheaves:

$$0 \rightarrow \mathcal{O}_X \xrightarrow{q} \mathcal{O}_X^* \xrightarrow{r} \mathcal{O}_X^* \rightarrow 1,$$

where  $q(x) := 1 + px$  for  $x \in \mathcal{O}_X$ , gives rise to the exact sequence  $H^1(\tilde{X}, \mathcal{O}_X^*) \rightarrow H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathcal{O}_X) = 0$ . Therefore, the invertible sheaf  $\mathcal{L} := \mathcal{O}_X(D)$  on  $X$  extends to an invertible sheaf  $\tilde{\mathcal{L}}$  on  $\tilde{X}$ . Let  $s \in H^0(X, \mathcal{L})$  be a section corresponding to the divisor  $D$ . Then lifting  $D$  is nothing but to extend the section  $s$  to a section  $\tilde{s} \in H^0(\tilde{X}, \tilde{\mathcal{L}})$ . The long exact sequence associated to  $0 \rightarrow \mathcal{L} \rightarrow \tilde{\mathcal{L}} \rightarrow \mathcal{L} \rightarrow 0$  shows that it suffices to prove  $H^1(X, \mathcal{L}) = 0$ .

Write  $D \sim aE + bF$ , where  $F$  is the fiber of  $f$ ,  $a \geq 0$  and  $b \geq am$ . We use induction on  $a$  to prove that  $H^1(X, \mathcal{O}_X(aE + bF)) = 0$  holds for any  $a \geq 0$  and  $b \geq am$ . When  $a = 0$ , we have  $H^1(X, \mathcal{O}_X(bF)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)) = 0$ . Assume  $a > 0$ . The exact sequence  $H^1(X, \mathcal{O}_X((a-1)E + bF)) \rightarrow H^1(X, \mathcal{O}_X(aE + bF)) \rightarrow H^1(E, \mathcal{O}_E(b-am))$  and the induction hypothesis conclude the argument.  $\square$

**Lemma 2.4.** *Let  $X$  be a smooth scheme over  $k$ ,  $D$  a reduced divisor on  $X$ , and  $Z \subset X$  a closed subscheme smooth over  $k$  of codimension  $s \geq 2$ . Let  $\pi : X' \rightarrow X$  be the blow-up of  $X$  along  $Z$  with the exceptional divisor  $E$ ,  $D' = \pi_*^{-1}D$  the strict transform of  $D$ . Assume that  $(X, D + Z)$  admits a mixed lifting to  $W_2(k)$ . Then  $(X', D' + E)$  admits a mixed lifting to  $W_2(k)$ .*

*Proof.* Let  $(\tilde{X}, \tilde{D} + \tilde{Z})$  be a mixed lifting of  $(X, D + Z)$  to  $W_2(k)$ . Then  $\tilde{Z} \subset \tilde{X}$  is a closed subscheme smooth over  $W_2(k)$  of codimension  $s \geq 2$ . Let  $\tilde{I}$  be the ideal sheaf of  $\tilde{Z}$  in  $\tilde{X}$ ,  $\tilde{\pi} : \tilde{X}' \rightarrow \tilde{X}$  the blow-up of  $\tilde{X}$  along  $\tilde{Z}$  with the exceptional divisor  $\tilde{E}$ , and  $\tilde{D}' = \tilde{\pi}_*^{-1}\tilde{D}$ . By [Ha77, Corollary II.7.15], we have the following commutative diagram:

$$\begin{array}{ccc} X'' & \hookrightarrow & \tilde{X}' \\ \pi' \downarrow & & \downarrow \tilde{\pi} \\ X & \hookrightarrow & \tilde{X} \end{array}$$

where  $\pi' : X'' \rightarrow X$  is the blow-up of  $X$  with respect to the ideal sheaf  $\tilde{I} \otimes_{W_2(k)} k = I$ , the ideal sheaf of  $Z$  in  $X$ . Hence  $X'' = X'$  and  $\pi' = \pi$ . Since  $\tilde{X}$  is smooth over  $W_2(k)$ , so is  $\tilde{X}'$ . Note that  $\tilde{X}' \times_{\text{Spec } W_2(k)} \text{Spec } k = \mathbf{Proj}(\oplus_i \tilde{I}^i) \times_{\text{Spec } W_2(k)} \text{Spec } k = \mathbf{Proj}(\oplus_i \tilde{I}^i \otimes_{W_2(k)} k) = \mathbf{Proj}(\oplus_i I^i) = X'$ , so  $\tilde{X}'$  is a lifting of  $X'$  to  $W_2(k)$ . It is easy to see that  $\tilde{D}' \times_{\text{Spec } W_2(k)} \text{Spec } k = D'$  and  $\tilde{E} \times_{\text{Spec } W_2(k)} \text{Spec } k = E$ , hence  $(X', D' + E)$  has a mixed lifting  $(\tilde{X}', \tilde{D}' + \tilde{E})$  to  $W_2(k)$ .  $\square$

**Definition 2.5.** Let  $X$  be a smooth projective surface, and  $D$  a reduced divisor on  $X$ .  $D$  is said to be suitable if there exists a birational morphism  $f : X \rightarrow X_{\min}$  such that

- (1)  $f$  is the composition of some  $(-1)$ -curve contractions,
- (2)  $X_{\min}$  is a relatively minimal model, and
- (3)  $D$  contains the exceptional locus  $\text{Exc}(f)$  of  $f$ .

**Proposition 2.6.** *Let  $X$  be a smooth projective rational surface over  $k$ ,  $D = \sum_{j=1}^r D_j$  a suitable simple normal crossing divisor on  $X$ . Then  $(X, D)$  admits a lifting to  $W_2(k)$ .*

*Proof.* If  $\rho(X) = 1$ , then  $X \cong \mathbb{P}_k^2$  and the conclusion is obvious. From now on, we may assume  $\rho(X) \geq 2$ . By assumption, there is a sequence of  $(-1)$ -curve contractions:

$$X = X_n \xrightarrow{(-1)} X_{n-1} \xrightarrow{(-1)} \cdots \xrightarrow{(-1)} X_1 \xrightarrow{(-1)} X_0,$$

where  $X_0$  is a Hirzebruch surface, say  $H_m$  with  $m \geq 0$ .

Let  $E_i \subset X_i$  be the corresponding  $(-1)$ -curves whose images are the smooth closed points  $P_{i-1} \in X_{i-1}$  ( $1 \leq i \leq n$ ),  $\pi_i : X \rightarrow X_i$  the induced morphisms ( $0 \leq i \leq n$ ), and  $E'_i = \pi_{i*}^{-1} E_i$  the strict transforms on  $X$  ( $1 \leq i \leq n$ ). By assumption,  $\sum_{i=1}^n E'_i$  is contained in  $D = \sum_{j=1}^r D_j$ . Let  $D^i = \pi_{i*} D$ ,  $0 \leq i \leq n-1$ . Then in general the irreducible components of  $D^0$  are neither smooth nor intersect transversally.

First of all, we assume  $P_i \in D^i$  for all  $0 \leq i \leq n-1$ . Then  $\pi_0 : D \subset X \rightarrow D^0 \subset X_0$  is a procedure consisting of a sequence of one point blow-ups such that the support of the total transform of  $D^0$  is equal to the support of  $D$ , which is simple normal crossing.

By Lemma 2.3,  $(X_0, D^0)$  has a mixed lifting  $(\widetilde{X}_0, \widetilde{D}^0)$  to  $W_2(k)$ . Let  $\eta : D^0 \hookrightarrow \widetilde{D}^0$  be the induced closed immersion, and let  $\widetilde{P}_0 = \eta(P_0) \in \widetilde{D}^0$ . If  $P_0 \in X_0$  is locally defined by equations  $x = x_0, y = y_0$ , then  $\widetilde{P}_0$  is locally defined by equations  $x = \widetilde{x}_0, y = \widetilde{y}_0$  with  $r(\widetilde{x}_0) = x_0, r(\widetilde{y}_0) = y_0$ , where  $x_0, y_0 \in k, \widetilde{x}_0, \widetilde{y}_0 \in W_2(k)$ . Therefore  $(X_0, D^0 + P_0)$  has a mixed lifting  $(\widetilde{X}_0, \widetilde{D}^0 + \widetilde{P}_0)$  to  $W_2(k)$ . By Lemma 2.4,  $(X_1, D^1)$  has a mixed lifting  $(\widetilde{X}_1, \widetilde{D}^1)$  to  $W_2(k)$ . We can repeat the same argument as above and use the induction on  $n$  to prove that  $(X, D)$  has a mixed lifting  $(\widetilde{X}, \widetilde{D})$  to  $W_2(k)$ , which is indeed a lifting of  $(X, D)$  to  $W_2(k)$ .

In general, if  $P_i \notin D^i$  for some  $i$ , then  $P_i$  is isolated from  $D^0$  (we denote the image of  $P_i$  in  $X_0$  by the same symbol), and we can further prove that  $(X_0, P_i)$  has a mixed lifting to  $W_2(k)$ , hence so does  $(X_i, D^i + P_i)$ . The rest is the same as above.  $\square$

### 3 Proof of the main theorem

The following vanishing result [KK, Corollary 2.2.5] is useful, which holds in arbitrary characteristic.

**Lemma 3.1.** *Let  $h : Y \rightarrow X$  be a proper birational morphism between normal surfaces with  $Y$  smooth and with exceptional locus  $E = \cup_{i=1}^s E_i$ . Let  $L$  be an integral divisor on  $Y$ ,  $0 \leq b_1, \dots, b_s < 1$  rational numbers, and  $N$  an  $h$ -nef  $\mathbb{Q}$ -divisor on  $Y$ . Assume  $L \equiv K_Y + \sum_{i=1}^s b_i E_i + N$ . Then  $R^1 h_* \mathcal{O}_Y(L) = 0$  holds.*

We can use Lemma 3.1 to show that the KLT surface singularity is rational in positive characteristic, while the general statement that the KLT singularity is rational in characteristic zero has been proved in [KM98, Theorem 5.22].

**Lemma 3.2.** *Let  $X$  be a normal proper surface, and  $B$  an effective  $\mathbb{Q}$ -divisor on  $X$  such that  $(X, B)$  is KLT. Then  $X$  has only rational singularities.*

*Proof.* Let  $h : Y \rightarrow X$  be the minimal resolution of  $X$ . Write  $K_Y \equiv h^*K_X + \sum_{i=1}^s a_i E_i$  with  $-1 < a_i \leq 0$  for all  $i$ , and  $h_*^{-1}B \equiv h^*B + \sum_{i=1}^s c_i E_i$  with  $c_i \leq 0$  for all  $i$ . Hence we have  $K_Y + h_*^{-1}B \equiv h^*(K_X + B) + \sum_{i=1}^s b_i E_i$  with  $b_i = a_i + c_i \leq 0$ . Since  $(X, B)$  is KLT,  $b_i > -1$  holds for all  $i$ . Since  $0 \equiv K_Y + \sum_{i=1}^s (-b_i) E_i + h_*^{-1}B - h^*(K_X + B)$ , by Lemma 3.1, we have  $R^1 h_* \mathcal{O}_Y = 0$ . It is easy to see that  $R^1 h_* \omega_Y = 0$  holds.  $\square$

*Proof of Theorem 1.4.* Take a log resolution  $h : Y \rightarrow X$  such that

- (1)  $Y$  is a smooth projective rational surface over  $k$ , and we can write  $K_Y + h_*^{-1}B \equiv h^*(K_X + B) + \sum_i a_i E_i$ , where  $E_i$  are the exceptional curves of  $h$  and  $a_i > -1$  for all  $i$ .
- (2)  $G = \text{Supp}(h_*^{-1}B) \cup \text{Exc}(h) \cup (\text{some self-intersection negative curves on } Y)$  is suitable and simple normal crossing.

Let  $D_Y = \lceil h^*D + \sum_i a_i E_i \rceil$ . Since  $\lceil \sum_i a_i E_i \rceil \geq 0$  is supported by  $\text{Exc}(h)$ , we have  $h_* \mathcal{O}_Y(D_Y) = \mathcal{O}_X(D)$  by the projection formula. Since  $\{h^*D + \sum_i a_i E_i\}$  is supported by  $\text{Exc}(h)$ , we can take  $0 < \delta_i \ll 1$  such that

$$(1) [h_*^{-1}B + \{h^*D + \sum_i a_i E_i\} + \sum_i \delta_i E_i] = 0.$$

(2)  $D_Y - (K_Y + h_*^{-1}B + \{h^*D + \sum_i a_i E_i\} + \sum_i \delta_i E_i) \equiv h^*(D - (K_X + B)) - \sum_i \delta_i E_i$  is ample.

Let  $B_Y = h_*^{-1}B + \{h^*D + \sum_i a_i E_i\} + \sum_i \delta_i E_i$ . Then  $H_Y = D_Y - (K_Y + B_Y)$  is ample,  $\text{Supp}(\langle H_Y \rangle) = \text{Supp}(B_Y)$  is simple normal crossing, and  $K_Y + \lceil H_Y \rceil = D_Y$ . Note that

$$D_Y \equiv K_Y + \{h^*D + \sum_i a_i E_i\} + h^*(D - (K_X + B)) + h_*^{-1}B.$$

By Lemma 3.1, we have  $R^1 h_* \mathcal{O}_Y(D_Y) = 0$ , hence  $H^1(Y, D_Y) = H^1(X, h_* \mathcal{O}_Y(D_Y)) = H^1(X, D)$ .

Since  $G$  is a suitable simple normal crossing divisor on the smooth rational surface  $Y$ , by Proposition 2.6,  $(Y, G)$  admits a lifting to  $W_2(k)$ . Since  $G$  contains  $\text{Supp}(\langle H_Y \rangle)$ , we have  $H^1(X, D) = H^1(Y, D_Y) = H^1(Y, K_Y + \lceil H_Y \rceil) = 0$  by Theorem 1.3.  $\square$

*Proof of Corollary 1.6.* It follows from the cone theorem [KK, 2.1.1 and 2.1.4] that the Kleiman-Mori cone  $\overline{NE}(X)$  is generated by rational curves. By Lemma 3.2,  $X$  has only rational singularities, therefore  $X$  is rational. The rest is due to Theorem 1.4.  $\square$

**Corollary 3.3.** *Let  $X$  be a weak log del Pezzo surface, and  $D$  a  $\mathbb{Q}$ -Cartier Weil divisor on  $X$  such that  $D - K_X$  is nef and big. Then  $H^1(X, D) = 0$  holds.*

*Proof.* Take an effective  $\mathbb{Q}$ -divisor  $B_1$  such that  $(X, B_1)$  is KLT and  $-(K_X + B_1)$  is ample. Then  $X$  is rational by the same argument as above. Take another effective  $\mathbb{Q}$ -divisor  $B_2$  such that  $(X, B_2)$  is KLT and  $D - (K_X + B_2)$  is ample. The rest is due to Theorem 1.4.  $\square$

*Proof of Corollary 1.7.* It follows from Theorem 1.4 or Corollary 3.3.  $\square$

**Corollary 3.4.** *Let  $X$  be a smooth projective rational surface,  $f : X \rightarrow \mathbb{P}^1$  a surjective projective morphism, and  $H$  an  $f$ -ample  $\mathbb{Q}$ -divisor on  $X$  such that the fractional part  $\langle H \rangle$  has simple normal crossing support. Then  $R^1 f_* \mathcal{O}_X(K_X + \lceil H \rceil) = 0$  holds.*

*Proof.* By assumption, there exists an  $m \in \mathbb{N}$  such that  $mH$  is integral and the natural morphism  $f^* f_* \mathcal{O}_X(mH) \rightarrow \mathcal{O}_X(mH)$  is surjective, which induces a closed immersion  $\varphi : X \rightarrow \mathbb{P}(f_* \mathcal{O}_X(mH))$  with  $mH = \varphi^* \mathcal{O}(1)$ . Therefore  $H$  is ample on  $X$ .

Let  $P$  be a general point in  $\mathbb{P}^1$ ,  $F = f^{-1}(P)$  the general fibre of  $f$ , and  $m$  a positive integer. Consider the Leray spectral sequence  $E_2^{ij} = H^i(\mathbb{P}^1, R^j f_* \mathcal{O}_X(K_X + \lceil H \rceil + mF)) \Rightarrow H^{i+j}(X, \mathcal{O}_X(K_X + \lceil H \rceil + mF))$ . By Serre vanishing,  $E_2^{ij} = 0$  holds for any  $i > 0$  and any  $m \gg 0$ . Therefore we have  $H^0(\mathbb{P}^1, R^1 f_* \mathcal{O}_X(K_X + \lceil H \rceil + mF)) = H^1(X, \mathcal{O}_X(K_X + \lceil H \rceil + mF)) = 0$  by Theorem 1.4. Note that  $R^1 f_* \mathcal{O}_X(K_X + \lceil H \rceil + mF) = R^1 f_* \mathcal{O}_X(K_X + \lceil H \rceil) \otimes \mathcal{O}_{\mathbb{P}^1}(m)$  is generated by global sections for  $m \gg 0$ , so we have  $R^1 f_* \mathcal{O}_X(K_X + \lceil H \rceil) = 0$ .  $\square$

*Proof of Corollary 1.9.* (1) We proceed a similar argument to the proof of Theorem 1.4 to obtain a log resolution  $h : Y \rightarrow X$  from a smooth projective rational surface  $Y$ , a divisor  $D_Y$  and a  $\mathbb{Q}$ -divisor  $B_Y$  on  $Y$ , such that  $f \circ h$  is projective,  $H_Y = D_Y - (K_Y + B_Y)$  is ample,  $[B_Y] = 0$  and  $\text{Supp}(B_Y)$  is simple normal crossing. Furthermore, we have  $R^1 h_* \mathcal{O}_Y(D_Y) = 0$  and  $h_* \mathcal{O}_Y(D_Y) = \mathcal{O}_X(D)$ .

Let  $g = f \circ h : Y \rightarrow \mathbb{P}^1$  be the induced morphism. It follows from Corollary 3.4 that  $R^1 f_* \mathcal{O}_X(D) = R^1 g_* \mathcal{O}_Y(D_Y) = R^1 g_* \mathcal{O}_Y(K_Y + \lceil H_Y \rceil) = 0$ . By the Leray spectral sequence and Theorem 1.4, we have  $H^1(\mathbb{P}^1, f_* \mathcal{O}_X(D)) = H^1(X, D) = 0$ .

(2) Since  $\mathcal{O}_X(D)$  is torsion free, so is  $f_* \mathcal{O}_X(D)$ . Hence  $f_* \mathcal{O}_X(D)$  is a locally free sheaf on  $\mathbb{P}^1$ . By Grothendieck's theorem (cf. [OSS80]),  $f_* \mathcal{O}_X(D)$  is a direct sum of invertible sheaves on  $\mathbb{P}^1$ :  $f_* \mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n)$ . Note that

$$\begin{aligned} H^1(X, D) &= H^1(\mathbb{P}^1, f_* \mathcal{O}_X(D)) = \bigoplus_i H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_i)), \\ f_* \mathcal{O}_X(D - f^* K_{\mathbb{P}^1}) &= f_* \mathcal{O}_X(D) \otimes \omega_{\mathbb{P}^1}^{-1} = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(d_i + 2), \end{aligned}$$

so the vanishing of  $H^1(X, D)$  implies the ampleness of  $f_* \mathcal{O}_X(D - f^* K_{\mathbb{P}^1})$ .  $\square$

## 4 Some remarks on the main results

First of all, we recall the following criterion for the liftability of log pairs.

**Lemma 4.1.** *Let  $X$  be a smooth variety, and  $D$  a simple normal crossing divisor on  $X$ . Then there is an obstruction  $o(X, D) \in \text{Ext}_{\mathcal{O}_X}^2(\Omega_X^1(\log D), \mathcal{O}_X) = H^2(X, \mathcal{T}_X(-\log D))$  to the liftability of  $(X, D)$  to  $W_2(k)$ , i.e.  $o(X, D) = 0$  if and only if  $(X, D)$  is liftable to  $W_2(k)$ .*

*Proof.* The case when  $D = \emptyset$  was verified directly in [Il96, Proposition 2.12]. For the general case, [EV92, Proposition 8.22] just showed that the isomorphisms of liftings of  $(X, D)$  over an open subset form a “torseur” under the group  $\text{Hom}_{\mathcal{O}_X}(\Omega_X^1(\log D), \mathcal{O}_X)$ , hence by a similar argument to that of [Il96, Proposition 2.12], we get the required obstruction  $o(X, D) \in \text{Ext}_{\mathcal{O}_X}^2(\Omega_X^1(\log D), \mathcal{O}_X) = H^2(X, \mathcal{T}_X(-\log D))$ . An alternative proof follows from a deep result in [DI87, 4.2.3]: the “gerbe”  $\text{rel}(X, D, W_2(k))$  of liftings of  $(X, D)$  is canonically equivalent to the “gerbe”  $\text{sc}(\tau_{\leq 1} F_* \Omega_X^\bullet(\log D))$  of splittings of the complex  $\tau_{\leq 1} F_* \Omega_X^\bullet(\log D)$ . Hence we have  $o(X, D) = \text{cl } \text{sc}(\tau_{\leq 1} F_* \Omega_X^\bullet(\log D)) \in \text{Ext}_{\mathcal{O}_X}^2(\Omega_X^1(\log D), \mathcal{O}_X)$  by [DI87, 3.2].  $\square$

Abelian varieties and complete intersections in  $\mathbb{P}_k^n$  are liftable to  $W_2(k)$ , which are nontrivial results of Grothendieck and Deligne (see [Il96, 7.11]). On the other hand, from Lemma 4.1, it follows easily that any smooth projective curve (or any log pair on it) is liftable to  $W_2(k)$ . Furthermore, we have the following consequence:



**Corollary 4.2.** *Let  $X$  be a smooth projective surface with  $\kappa(X) < 0$ . Then  $X$  is liftable to  $W_2(k)$ .*

*Proof.* If  $X \cong \mathbb{P}_k^2$  then the conclusion is obvious. So we may assume that there is a fibration  $f : X \rightarrow C$  over a smooth projective curve  $C$  with a general fiber  $F \cong \mathbb{P}^1$ . By Lemma 4.1 and Serre duality, it suffices to show  $H^0(X, \Omega_X^1 \otimes \omega_X) = 0$ . Suppose to the contrary that  $H^0(X, \Omega_X^1 \otimes \omega_X) \neq 0$  holds, then we can take a section  $0 \neq s \in H^0(X, \Omega_X^1 \otimes \omega_X)$  such that  $0 \neq s|_F \in H^0(F, \Omega_X^1 \otimes \omega_X|_F)$ . Let  $\mathcal{I} = \mathcal{O}_X(-F)$  be the ideal sheaf of  $F$  in  $X$ . Then we have  $\mathcal{I}/\mathcal{I}^2 = \mathcal{O}_X(-F)|_F \cong \mathcal{O}_F$ . By adjunction formula, we have  $\omega_X|_F \cong \omega_F \cong \mathcal{O}_F(-2)$ . Tensoring the following exact sequence with  $\omega_X|_F$ :

$$0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_X^1|_F \rightarrow \omega_F \rightarrow 0,$$

we have the exact sequence:

$$0 \rightarrow \mathcal{O}_F(-2) \rightarrow \Omega_X^1 \otimes \omega_X|_F \rightarrow \mathcal{O}_F(-4) \rightarrow 0.$$

Taking the long exact sequence of cohomology groups, we have  $H^0(F, \Omega_X^1 \otimes \omega_X|_F) = 0$ , which is a contradiction.  $\square$

The main technical result in this paper is Proposition 2.6, which shows that certain log pairs on smooth rational surfaces are liftable to  $W_2(k)$ . The proof of Proposition 2.6 is proceeded by induction via Lemma 2.4, however the initial step, i.e. Lemma 2.3, is proved by a argument, which depends on the geometric properties of Hirzebruch surfaces. Therefore it seems impossible to generalize Proposition 2.6 to general surfaces. In fact, Proposition 2.6 fails even for certain ruled surfaces, which is described in Corollary 1.10, since there exist counterexamples to the Kawamata-Viehweg vanishing on those ruled surfaces.

*Proof of Corollary 1.10.* We use the same notation and construction as in [Xie07, Theorem 3.1]. Therefore, there are a  $\mathbb{P}^1$ -bundle  $f : X \rightarrow C$  and an ample  $\mathbb{Q}$ -divisor  $H$  on  $X$  with  $\text{Supp}(\langle H \rangle) = C'$  and  $H^1(X, K_X + \lceil H \rceil) \neq 0$ , where  $C' \subset X$  is a smooth curve and  $f|_{C'} : C' \rightarrow C$  is the  $k$ -linear Frobenius morphism. By Theorem 1.3,  $(X, C')$  cannot be lifted to  $W_2(k)$ .  $\square$

Note that Corollary 1.10 means  $0 \neq o(X, C') \in H^2(X, \mathcal{T}_X(-\log C'))$ , while the  $\mathbb{P}^1$ -bundle  $X$  itself is liftable to  $W_2(k)$  by Corollary 4.2.

Finally, we give some remarks on Theorem 1.4.

*Remark 4.3.* (1) By a standard argument via Kodaira's lemma, Theorem 1.4 gives rise to the Kawamata-Viehweg vanishing theorem for nef and big  $\mathbb{Q}$ -divisors on smooth rational surfaces, which may be useful in practice.

(†) Let  $X$  be a smooth proper rational surface, and  $L$  a nef and big  $\mathbb{Q}$ -divisor on  $X$ , such that the fractional part  $\langle L \rangle$  has simple normal crossing support. Then  $H^1(X, K_X + \lceil L \rceil) = 0$  holds.

(2) The following Kodaira-Ramanujam vanishing theorem [Ra72] is a special case of the Kawamata-Viehweg vanishing theorem for nef and big integral divisors on smooth surfaces.

( $\dagger$ ) Let  $X$  be a smooth projective surface over an algebraically closed field  $k$  with  $\text{char}(k) = 0$ , and  $L$  a nef and big integral divisor on  $X$ . Then  $H^1(X, K_X + L) = 0$  holds.

By a result of Raynaud [DI87, Corollaire 2.8] and Corollary 4.2, the Kodaira-Ramanujam vanishing theorem holds on all smooth projective surfaces with negative Kodaira dimension in positive characteristic, while among those surfaces, there exist counterexamples to the Kawamata-Viehweg vanishing theorem for nef and big  $\mathbb{Q}$ -divisors (see [Xie07, Theorem 3.1]). This observation shows that there is a significant difference between the  $\mathbb{Q}$ -divisor version and the integral divisor version of the Kawamata-Viehweg vanishing theorem in positive characteristic.

## References

- [DI87] P. Deligne, L. Illusie, Relèvements modulo  $p^2$  et décomposition du complexe de de Rham, *Invent. Math.*, **89** (1987), 247–270.
- [EV92] H. Esnault, E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. **20**, Birkhäuser, 1992.
- [Ha98] N. Hara, A characterization of rational singularities in terms of injectivity of Frobenius maps, *Amer. J. Math.*, **120** (1998), 981–996.
- [Ha77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.
- [Il96] L. Illusie, Frobenius et dégénérescence de Hodge, in J. Bertin, J.-P. Demailly, L. Illusie and C. Peters, *Introduction à la Théorie de Hodge*, Panoramas et Synthèses, vol. **3**, Société de Mathématiques de France, Marseilles, 113–168, 1996.
- [Ka82] Y. Kawamata, A generalization of Kodaira-Ramanujam’s vanishing theorem, *Math. Ann.*, **261** (1982), 43–46.
- [Ka00] Y. Kawamata, On effective non-vanishing and base-point-freeness, *Asian J. Math.*, **4** (2000), 173–182.
- [KMM87] Y. Kawamata, K. Matsuda, K. Matsuki, Introduction to the minimal model problem, Alg. Geom. Sendai 1985, *Adv. Stud. Pure Math.*, **10** (1987), 283–360.
- [Ko95] J. Kollár, *Shafarevich maps and automorphic forms*, Princeton Univ. Press, 1995.
- [KK] J. Kollár, S. Kovács, Birational geometry of log surfaces, preprint, which can be found at <http://www.math.washington.edu/~kovacs/pdf/BiratLogSurf.pdf>.
- [KM98] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., vol. **134**, 1998.
- [LR97] N. Lauritzen, A. Rao, Elementary counterexamples to Kodaira vanishing in prime characteristic, *Proc. Indian Acad. Sci. Math. Sci.*, **107** (1997), 21–25.
- [MO05] K. Matsuki, M. Olsson, Kawamata-Viehweg vanishing as Kodaira vanishing for stacks, *Math. Res. Lett.*, **12** (2005), 207–217.
- [OSS80] C. Okonek, M. Schneider, H. Spindler, *Vector bundles on complex projective space*, Progress in Math., vol. **3**, 1980.
- [Ra72] C. P. Ramanujam, Remarks on the Kodaira vanishing theorem, *J. Indian. Math. Soc.*, **36** (1972), 41–51.
- [Re94] M. Reid, Nonnormal del Pezzo surfaces, *Publ. Res. Inst. Math. Sci.*, **30** (1994), 695–727.

- [Sc07] S. Schröer, Weak del Pezzo surfaces with irregularity, *Tôhoku Math. J.*, **59** (2007), 293-322.
- [Se62] J.-P. Serre, *Corps locaux*, Hermann, 1962.
- [SB97] N. Shepherd-Barron, Fano threefolds in positive characteristic, *Compositio Math.*, **105** (1997), 237-265.
- [Ta72] H. Tango, On the behavior of cohomology classes of vector bundles under the Frobenius map (in Japanese), *Res. Inst. Math. Sci., Kôkyûroku*, **144** (1972), 93-102.
- [Vi82] E. Viehweg, Vanishing theorems, *J. Reine Angew. Math.*, **335** (1982), 1-8.
- [Xie06] Q. Xie, Effective non-vanishing for algebraic surfaces in positive characteristic, *J. Algebra*, **305** (2006), 1111-1127.
- [Xie07] Q. Xie, Counterexamples to the Kawamata-Viehweg vanishing on ruled surfaces in positive characteristic, preprint, math.AG/0702554.

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